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On Asymptotic Behavior of Multilinear Eigenvalue Statistics of Random Matrices

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Abstract We prove the Law of Large Numbers and the Central Limit Theorem for analogs of U- and V- (von Mises) statistics of eigenvalues of random matrices as their size tends to infinity. We show first that for a certain class of test functions (kernels), determining the statistics, the validity of these limiting laws reduces to the validity of analogous facts for certain linear eigenvalue statistics. We then check the conditions of the reduction statements for several most known ensembles of random matrices. The reduction phenomenon is well known in statistics, dealing with i.i.d. random variables. It is of interest that an analogous phenomenon is also the case for random matrices, whose eigenvalues are strongly dependent even if the entries of matrices are independent.

Keywords Random matrices · Multilinear eigenvalue statistics · Central limit theorem

1 Introduction

In recent decades there has been a considerable activity in studying asymptotic properties of linear eigenvalue statistics

$$\mathcal{N}_n[\varphi] = \sum_{l=1}^{n} \varphi(\lambda_l^{(n)}) \tag{1}$$

for various classes of random symmetric or hermitian matrices. We denote in (1) $\{\lambda_l^{(n)}\}_{l=1}^n$ eigenvalues of $n \times n$ random matrix M, assuming that they are indexed in the non-decreasing order, and $\varphi : \mathbb{R} \to \mathbb{R}$ a function, called often the test function (or the kernel of statistic). It has been proved that for a rather broad class of random matrix ensembles and any bounded and continuous test function $n^{-1}\mathcal{N}_n[\varphi]$ converges either in probability or with probability 1

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to a non-random limit

$$\int_{\mathbb{R}} \varphi(\lambda) N(d\lambda), \tag{2}$$

where N is a probability measure known as the Integrated Density of States of the ensemble. In other words, if for any $\Delta \subset \mathbb{R}$

$$N_n(\Delta) = \sharp\{\lambda_l^{(n)} \in \Delta,\ l = 1, \ldots, n\}/n, \tag{3}$$

i.e., N_n is the Normalized Counting Measure of eigenvalues of M, then N_n converges either in probability or with probability 1 to a non-random measure N (see [2, 10, 11, 22, 24] for results and references).

This can be viewed as an analog of the Law of Large Numbers for linear eigenvalue statistics. Note that the eigenvalues of random matrices are strongly dependent random variables even in the case, when the entries of matrices are independent (modulo the symmetry condition of course). Nevertheless, the analogs of the Central Limit Theorem (CLT) for linear eigenvalue statistics have also been found in various instances (see e.g. [1, 3, 6, 8, 9, 12–15, 25–27]) although the situation with CLT in the random matrix theory is more subtle up to certain cases, where other limiting laws of fluctuations emerge (see e.g. [23]). The reason is that due to mention above strong dependence of eigenvalues the variance of linear eigenvalue statistics of many random matrices does not grow with n, and the mathematical mechanism of the limiting Gaussian law is more subtle than just the "collective effect" of large number of small random and independent random variables.

Note now that the linear statistic (1) is an analog of an additive observable of statistical mechanics and condensed matter theory, where the binary, ternary, etc. observables are also of considerable interest. Their analogs exist also in statistics and known as U-statistics. Thus, it is natural to consider similar objects for eigenvalues of random matrices, defined via a bounded symmetric $\varphi : \mathbb{R}^p \to \mathbb{R}$, $p \geq 1$ as

$$\mathcal{U}_{pn}[\varphi] = \sum_{1 \leq l_1 < l_2 < \cdots < l_p \leq n} \varphi\left(\lambda_{l_1}^{(n)}, \ldots, \lambda_{l_p}^{(n)}\right). \tag{4}$$

One can also consider a bit different object that we call the multilinear statistic and that is known in statistics as V- or von-Mises statistic

$$\mathcal{N}_{pn}[\varphi] = \sum_{l_1, \ldots, l_p = 1}^{n} \varphi\left(\lambda_{l_1}^{(n)}, \ldots, \lambda_{l_p}^{(n)}\right), \tag{5}$$

that can also can be written as the function of the pth tensor power of M:

$$\mathcal{N}_{pn}[\varphi] = \operatorname{Tr} \varphi(M^{\otimes p}). \tag{6}$$

As was mentioned above, the eigenvalues of random matrices are strongly dependent random variables. Thus, the methods, developed in statistics for the analysis of U-statistics and von-Mises statistics (analogs of (4) and (5) with i.i.d. random variables instead of $\{\lambda_l^{(n)}\}_{l=1}^n$) and based mainly on the martingale theory ideas [16, 17], are not directly applicable for random matrices. Nevertheless, we show below that the analysis of multilinear statistics essentially reduces to that of linear statistics, similarly to the situation in statistics. This allows us to use the known results for linear eigenvalue statistics to prove analogs of the Law of

Large Numbers and the Central Limit Theorem for multilinear statistics of several important classes of random matrices.

The paper is organized as follows. In Sect. 2 we study the analogs of the Law of Large Numbers and in Sect. 3 the analogs of the Central Limit Theorem. In both cases we prove first a general statement, allowing us to reduce the case $p \geq 2$ in (5) to the case $p = 1$, and then we check the validity of the hypotheses of the reduction statements for various random matrices.

2 Law of Large Numbers for Multilinear Statistics

We give here the assertions, corresponding to the Strong Law of Large Numbers (convergence with probability 1) and the Weak Law of Large Numbers (convergence in probability). The results are rather simple consequences of those for linear statistics.

Theorem 1 *Let M be an $n \times n$ real symmetric or hermitian random matrix. We have*:

(i) *if the Normalized Counting Measure of eigenvalues of M (see (3)) converges weakly with probability* 1 *to a non-random probability measure N*:

$$\lim_{n\to\infty} N_n = N, \tag{7}$$

then for any bounded, continuous, and symmetric $\varphi : \mathbb{R}^p \to \mathbb{R}$ the normalized statistics $n^{-p}\mathcal{N}_{pn}[\varphi]$ and $n^{-p}\mathcal{U}_{pn}[\varphi]$, where $\mathcal{N}_{pn}$ and $\mathcal{U}_{pn}$ are given by (5) *and* (4), *converge with probability* 1 *to the non-random limit*:

$$\lim_{n\to\infty} n^{-p}\mathcal{N}_{pn}[\varphi] = p! \lim_{n\to\infty} n^{-p}\mathcal{U}_{pn}[\varphi] = \mathcal{L}_p[\varphi], \tag{8}$$

where

$$\mathcal{L}_p[\varphi] = \int_{\mathbb{R}^p} \varphi(\lambda_1, \dots, \lambda_p) N(d\lambda_1) \dots N(d\lambda_p); \tag{9}$$

(ii) *if for any $\Delta \subset \mathbb{R}$*

$$\lim_{n\to\infty} \mathbf{E}\{|N_n(\Delta) - N(\Delta)|\} = 0, \tag{10}$$

then for any bounded, continuous, and symmetric $\varphi : \mathbb{R}^p \to \mathbb{R}$

$$\lim_{n\to\infty} \mathbf{E}\{|n^{-p}\mathcal{N}_{pn}[\varphi] - \mathcal{L}_p[\varphi]|\} = \lim_{n\to\infty} \mathbf{E}\{|n^{-p}\mathcal{U}_{pn}[\varphi] - (p!)^{-1}\mathcal{L}_p[\varphi]|\} = 0. \tag{11}$$

Proof (i) It is known that if a sequence of measures $\{m_n\}$ on $\mathbb{R}$ converges weakly to a probability measure m on $\mathbb{R}$, then the sequence of p-fold product measures $\{m_n^{\otimes p}\}$ on $\mathbb{R}^p$ converges weakly to the p-fold product $m^{\otimes p}$ of the measures m (see e.g. [4], Theorem 3.2). This fact and the hypothesis of assertion (i) imply its validity for $\mathcal{N}_{pn}$. To prove the assertion for $\mathcal{U}_{pn}$ we note that since φ is symmetric in its arguments

$$n^{-p}((p!)^{-1}\mathcal{N}_{pn}[\varphi] - \mathcal{U}_{pn}[\varphi])$$

is the finite number (depending only on p) sums of the type (5) but having at least two coinciding indices. Since the summands are bounded by $\sup_{\lambda\in\mathbb{R}^p}|\varphi(\lambda)|$ and their number in every sum is n^{p-1} at most, we obtain the bound

$$n^{-p}\left|(p!)^{-1}\mathcal{N}_{pn}[\varphi]-\mathcal{U}_{pn}[\varphi]\right|\le C_p n^{-1}\sup_{\lambda\in\mathbb{R}^p}|\varphi(\lambda)|,$$

where C_p depends only on p. The bound and the validity of assertion (i) for $\mathcal{N}_{pn}$ imply its validity for $\mathcal{U}_{pn}$.

(ii) To prove the assertion we first choose $\varepsilon>0$ and a cube $C_A\subset\mathbb{R}^p$, centered in the origin, having the side length A. Let now

$$C_A=\bigcup_{\alpha=1}^{\nu}C_\alpha$$

be a partition of C_A in which C_α's are so small that the variation of φ in every C_α does not exceed ε. Then we have

$$n^{-p}\mathcal{N}_{pn}[\varphi]\le\sum_{\alpha=1}^{\nu}(\varphi(\lambda_\alpha)+\varepsilon)N_n^{\otimes p}(C_\alpha)+\sup_{\lambda\in\mathbb{R}^p}|\varphi(\lambda)|N_n^{\otimes p}(\mathbb{R}\backslash C_A)$$

and

$$n^{-p}\mathcal{N}_{pn}[\varphi]\ge\sum_{\alpha=1}^{\nu}(\varphi(\lambda_\alpha)-\varepsilon)N_n^{\otimes p}(C_\alpha)-\sup_{\lambda\in\mathbb{R}^p}|\varphi(\lambda)|N_n^{\otimes p}(\mathbb{R}\backslash C_A),$$

where $\lambda_\alpha\in C_\alpha$. Writing analogous inequalities for $\mathcal{L}_p[\varphi]$ and subtracting them from the above, we obtain that

$$\begin{aligned}\mathbf{E}\{|n^{-p}\mathcal{N}_{pn}[\varphi]-\mathcal{L}_p[\varphi]|\}&\le 2\varepsilon\sum_{\alpha=1}^{\nu}\mathbf{E}\{|N_n^{\otimes p}(C_\alpha)-N^{\otimes p}(C_\alpha)|\}\\&\quad+\sup_{\lambda\in\mathbb{R}^p}|\varphi(\lambda)|(\mathbf{E}\{N_n^{\otimes p}(\mathbb{R}\backslash C_A)\}+N^{\otimes p}(\mathbb{R}\backslash C_A)).\end{aligned}$$

Passing subsequently in this bound to the limits $n\to\infty$ and $A\to\infty$ and taking into account (10), we obtain assertion (ii) of the theorem, concerning $\mathcal{N}_{pn}$. An analogous assertion for $\mathcal{U}_{pn}$ follows from the argument similar to that used in the proof of assertion (i). □

We will list now several classes of random matrices for which the hypotheses of Theorem 1 are known to be valid:

(i) Classical Ensembles: GOE, GUE, Wishart, Laguerre, etc. (see e.g. [10, 19] for their definitions and properties), as well as their deformed versions $H^{(0)}+M$, where $H^{(0)}$ is non-random (or random but independent of M) and such that its Normalized Counting Measure converges weakly to a probability measure of compact support. For these ensembles the limiting relation (7) is known since long time (see e.g. [2, 11, 22] and references therein). In particular, for $H^{(0)}=0$ their limiting measures are absolutely continuous (under certain conditions for the Wishart and Laguerre), and the corresponding densities are

$$(2\pi)^{-1}\sqrt{4-\lambda^2}\mathbf{1}_{\{|\lambda|\le 2\}},\tag{12}$$

and

$$(2\pi\lambda)^{-1}\sqrt{(a_+-\lambda)(\lambda-a_-)}\mathbf{1}_{\{\lambda\in[a_-,a_+]\}}, \tag{13}$$

where $c \geq 1,\ a_\pm = (1 \pm \sqrt{c})^2$.

(ii) Wigner Ensembles, where $M = n^{-1/2}W$, $W = \{W_{jk}\}_{j,k=1}^n$, $W_{jk} = W_{kj}$ in the real symmetric case, $W_{jk} = \overline{W}_{kj}$ in the hermitian case, and $\{W_{jk}\}_{1\leq j\leq k\leq\infty}$ is the infinite collection of i.i.d. random variables of zero mean and variance 1, as well as their "deformed" versions. The limiting measure exists and is the same as for the Gaussian Ensembles (12).

(iii) The sample covariance matrices $M = n^{-1}X^*X$, as well as their "deformed" versions, where "*" denotes the matrix transposition in the real symmetric case and the hermitian conjugation in the hermitian case, and $X = \{X_{\alpha j}\}_{\alpha,j=1}^{m,n}$ is the $m \times n$ random matrix whose entries are taken from the infinite collection $\{X_{\alpha j}\}_{\alpha,j=1}^{\infty}$ of i.i.d. (real or complex) random variables, and $\lim_{n\to\infty} m/n = c \in [0,\infty)$. The limiting measure coincides with that for the Wishart and Laguerre Ensembles (13). For the validity of (7) in these cases see again [2, 11] and references therein.

(iv) The "triangular array" versions of the Wigner and the sample covariance random matrices, where $W = \{W_{jk}^{(n)}\}_{j,k=1}^n$ and $X = \{X_{\alpha j}^{(n)}\}_{\alpha,j=1}^{m,n}$, i.e., the entries are independent (modulo symmetry) random variables, whose probability law depends now on j, k, and n, and we do not assume that the entries for all n are defined on the same probability space. In these cases one has to use the convergence in probability (or its a bit stronger version (10)), and for its validity we refer to [2, 11, 20, 21] and references therein.

(v) Real symmetric and hermitian matrices, whose probability law is

$$Z_{n\beta}^{-1}\exp\left\{-n\beta\,\mathrm{Tr}\,V(M)/2\right\}d_\beta M, \tag{14}$$

where $\beta = 1$ for real symmetric matrices, $\beta = 2$ for hermitian matrices, $V : \mathbb{R} \to \mathbb{R}_+$ is locally Lipshitz function with the power $\alpha \in (0,1)$ in the corresponding inequality,

$$V(\lambda) \geq (2+\varepsilon)\log(1+|\lambda|), \quad |\lambda| \geq L \tag{15}$$

for some $\varepsilon > 0$ and $L < \infty$, $Z_{n\beta}$ is the normalization constant, and

$$d_1M = \prod_{1\leq j\leq k\leq n} dM_{jk}, \qquad d_2M = \prod_{j=1}^n dM_{jj} \prod_{1\leq j<k\leq n} d\Re M_{jk}d\Im M_{jk}. \tag{16}$$

In this case the validity of (10) is proved in [5, 13, 24]. The corresponding limiting measure N is a unique minimizer of a certain variational problem and has a compact support.

3 Fluctuations of Multilinear Statistics

In this section we study analogs of the CLT for statistics (4) and (5) of eigenvalues of various random matrices. The following theorem establishes the stochastic equivalence of centered multilinear statistics and certain linear statistics.

Theorem 2 *Let M be an $n \times n$ real symmetric or hermitian random matrix. Assume that*

(i) *the measure $\overline{N}_n = \mathbf{E}\{N_n\}$, where N_n is defined in (3), converges weakly to a probability measure N:*

$$\lim_{n\to\infty} \overline{N}_n = N, \tag{17}$$

(ii) *for*

$$u_n(t) = \mathrm{Tr}\, e^{itM} = n \int_{\mathbb{R}} e^{i\lambda t} N_n(d\lambda), \tag{18}$$

we have

$$\mathbf{Var}\{u_n(t)\} \le C(t), \tag{19}$$

where $C(t)$ is an n-independent polynomial in $|t|$ with positive coefficients.

Then for any symmetric and integrable $\varphi : \mathbb{R}^p \to \mathbb{R}$, $p \ge 2$, such that its Fourier transform

$$\widehat{\varphi}(t) = \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} e^{-i(x,t)} \varphi(x) d^p x \tag{20}$$

satisfies the condition

$$\int_{\mathbb{R}^p} C^{1/2}(t_1) C^{1/2}(t_2) |\widehat{\varphi}(t)| d^p t < \infty, \tag{21}$$

we have uniformly in $x \in \mathbb{R}$, varying in any finite interval,

$$\lim_{n\to\infty} \left(\mathbf{E}\left\{e^{ixn^{-p+1}\mathcal{N}^{\circ}_{pn}[\varphi]}\right\} - \mathbf{E}\left\{e^{ix\mathcal{N}^{\circ}_{1n}[\varphi^*_p]}\right\}\right) = 0, \tag{22}$$

and

$$\lim_{n\to\infty} \left(\mathbf{E}\left\{e^{ixn^{-p+1}\mathcal{U}^{\circ}_{pn}[\varphi]}\right\} - \mathbf{E}\left\{e^{ix\mathcal{N}^{\circ}_{1n}[(p!)^{-1}\varphi^*_p]}\right\}\right) = 0, \tag{23}$$

where

$$\mathcal{N}^{\circ}_{pn}[\varphi] = \mathcal{N}_{pn}[\varphi] - \mathbf{E}\{\mathcal{N}_{pn}[\varphi]\}, \qquad \mathcal{U}^{\circ}_{pn}[\varphi] = \mathcal{U}_{pn}[\varphi] - \mathbf{E}\{\mathcal{U}_{pn}[\varphi]\},$$

and $\varphi^*_p : \mathbb{R} \to \mathbb{R}$ is

$$\varphi^*_p(\lambda) = p \int_{\mathbb{R}^{p-1}} \varphi(\lambda, \lambda_2, \dots, \lambda_p) N(d\lambda_2) \dots N(d\lambda_p), \quad p \ge 1. \tag{24}$$

Proof It follows from the inequality $|e^{ia} - e^{ib}| \le |a - b|$, $a, b \in \mathbb{R}$ that

$$\begin{aligned} &\left|\mathbf{E}\left\{e^{ixn^{-p+1}\mathcal{N}^{\circ}_{pn}[\varphi]}\right\} - \mathbf{E}\left\{e^{ix\mathcal{N}^{\circ}_{1n}[\varphi^*_p]}\right\}\right| \\ &\quad \le |x| \mathbf{E}\left\{\left|n^{-p+1}\mathcal{N}^{\circ}_{pn}[\varphi] - \mathcal{N}^{\circ}_{1n}[\varphi^*_p]\right|\right\}. \end{aligned} \tag{25}$$

Thus it suffices to prove that the expectation on the right vanishes as $n \to \infty$.

We have from (5) and (20):

$$\mathcal{N}_{pn}[\varphi] = \int_{\mathbb{R}^p} \widehat{\varphi}(t) \prod_{q=1}^{p} u_n(t_q) d^p t, \tag{26}$$

where u_n is defined in (18). Writing

$$u_n = u_n^{\circ} + \overline{u}_n, \qquad \overline{u}_n = \mathbf{E}\{u_n\},$$

and using the symmetry of $\widehat{\varphi}$, we present (26) as

$$\mathcal{N}_{pn}[\varphi]=\sum_{q=0}^{p}\binom{p}{q}\int_{\mathbb{R}^p}\widehat{\varphi}(t)u_n^{\circ}(t_1)\dots u_n^{\circ}(t_q)\overline{u}_n(t_{q+1})\dots\overline{u}_n(t_p)d^pt. \tag{27}$$

Applying to the identity the operation of expectation and subtracting the result from (27), we obtain that

$$\mathcal{N}_{pn}^{\circ}[\varphi]=p\int_{\mathbb{R}^p}\widehat{\varphi}(t)u_n^{\circ}(t_1)\prod_{q=2}^{p}\overline{u}_n(t_q)d^pt$$
$$+\sum_{q=2}^{p}\binom{p}{q}\int_{\mathbb{R}^p}\widehat{\varphi}(t)\left(\prod_{r=1}^{q}u_n^{\circ}(t_r)-\mathbf{E}\Big\{\prod_{r=1}^{q}u_n^{\circ}(t_r)\Big\}\right)\prod_{r=q+1}^{p}\overline{u}_n(t_r)d^pt. \tag{28}$$

Denoting

$$v_n(t)=n^{-1}u_n(t)=n^{-1}\operatorname{Tr}e^{itM},\qquad \overline{v}_n(t)=n^{-1}\overline{u}_n(t),$$

and taking into account the inequalities

$$|v_n(t)|\le 1,\qquad |\overline{v}_n(t)|\le 1,\quad \forall t\in\mathbb{R}, \tag{29}$$

following from the unitarity of e^{itM} in (18), we obtain from (28)

$$\mathbf{E}\left\{\left|n^{-p+1}\mathcal{N}_{pn}^{\circ}[\varphi]-\int_{\mathbb{R}^p}p\widehat{\varphi}(t)u_n^{\circ}(t_1)\prod_{q=2}^{p}\overline{v}_n(t_q)d^pt\right|\right\}$$
$$\le n^{-1}\sum_{q=2}^{p}2^{q-1}\binom{p}{q}\int_{\mathbb{R}^p}|\widehat{\varphi}(t)|\mathbf{E}\{|u_n^{\circ}(t_1)u_n^{\circ}(t_2)|\}d^pt, \tag{30}$$

and then (19) and the Schwarz inequality imply

$$\mathbf{E}\left\{\left|n^{-p+1}\mathcal{N}_{pn}^{\circ}[\varphi]-\int_{\mathbb{R}^p}p\widehat{\varphi}(t)u_n^{\circ}(t_1)\prod_{q=2}^{p}\overline{v}_n(t_q)d^pt\right|\right\}$$
$$\le 3^pn^{-1}\int_{\mathbb{R}^p}|\widehat{\varphi}(t)|C^{1/2}(t_1)C^{1/2}(t_2)d^pt. \tag{31}$$

Denote v the Fourier transform of N of (17). It follows from (17) that

$$\lim_{n\to\infty}\overline{v}_n(t)=v(t)$$

uniformly in t on any compact set of $\mathbb{R}$. Thus $\prod_{q=2}^{p}\overline{v}_n(t_q)$ converges to $\prod_{q=2}^{p}v(t_q)$ uniformly on any compact set of $\mathbb{R}^{p-1}$, and taking into account (19) and (21), we conclude that the error of replacing in (31) $\prod_{q=2}^{p}\overline{v}_n(t_q)$ by $\prod_{q=2}^{p}v(t_q)$ vanishes as $n\to\infty$.

Note now that according to (18) and the spectral theorem

$$\int_{\mathbb{R}^p}p\widehat{\varphi}(t)u_n^{\circ}(t_1)\prod_{q=2}^{p}v(t_q)d^pt=\int_{\mathbb{R}}\widehat{\varphi}_p(t_1)u_n^{\circ}(t_1)dt_1$$
$$=\int_{\mathbb{R}}\varphi_p^{*}(\lambda_1)nN_n^{\circ}(d\lambda_1)=:\mathcal{N}_{1n}^{\circ}[\varphi_p^{*}],$$

where φ_p^* is the inverse Fourier transform of

$$\widehat{\varphi}_p(t_1) = p\int_{\mathbb{R}^{p-1}} \widehat{\varphi}(t_1, t_2, \ldots, t_p)\prod_{q=2}^{p} v(t_q)dt_2\ldots dt_p.$$

It follows from (20) that φ_p^* coincides with the r.h.s. of (24). This proves (22). As for (23) we consider the simplest case $p=2$, containing already the essence of general case of an arbitrary p.

We have from (4) and (5) that

$$n^{-1}\mathcal{U}_{2n}[\varphi] = (2n)^{-1}\mathcal{N}_{2n}[\varphi] - (2n)^{-1}\int_{\mathbb{R}}\varphi(\lambda,\lambda)\mathcal{N}_n(d\lambda).$$

Now an argument similar to that in the proof of (31) yields

$$\begin{aligned}&\mathbf{E}\left\{\left|n^{-1}\mathcal{U}_{2n}^{\circ}[\varphi] - \mathcal{N}_{1n}^{\circ}[2^{-1}\varphi_2^*]\right|\right\}\\ &\quad\le 2^{-1}\mathbf{E}\left\{\left|n^{-1}\mathcal{N}_{2n}^{\circ}[\varphi] - \mathcal{N}_{1n}^{\circ}[\varphi_2^*]\right|\right\} + (2n)^{-1}\int_{\mathbb{R}^2}|\widehat{\varphi}(t_1,t_2)|\mathbf{E}\{|u_n^{\circ}(t_1+t_2)|\}dt_1dt_2.\end{aligned}$$

In view of (21) the second term on the right is bounded by

$$(2n)^{-1}\int_{\mathbb{R}^2} C^{1/2}(t_1+t_2)|\widehat{\varphi}(t_1,t_2)|dt_1dt_2.$$

Since C is a polynomial in $|t|$ with positive coefficients, it admits the bound

$$C(t_1+t_2) \le A(C(t_1) + C(t_2)),$$

where A depends only on the degree of C. This, (21), and (22) imply (23). □

Remark 1 The same argument can be used to prove the CLT for multilinear (U- and V-) statistics of i.i.d. random variables $\{\xi_l\}_{l\ge1}$. In this case (17) is just the weak Law of Large Numbers for sums of i.i.d. random variables and (19) is valid with $C(t) = const$. Thus, Theorem 2 implies the stochastic equivalence of centered multilinear statistics for i.i.d. random variables and test functions with integrable Fourier transform and the centered linear statistics for the same random variables and test functions φ_p^* of (24). Since φ_p^* is bounded in this case and since the CLT for linear statistics is valid for such test functions, we conclude that the CLT for multilinear eigenvalue statistics of i.i.d. random variables is valid for test functions with integrable Fourier transform. One can then extend the CLT for wider classes of test functions, up to those, satisfying the condition $\mathbf{Var}\{\varphi_p^*(\xi_1)\} \in (0,\infty)$, usual in statistics and resulting form the martingale techniques (see e.g. [17], Sect. 3.1). The extensions can be obtained by a standard approximation procedure (see e.g. its version in item (i) below, treating the Gaussian Ensembles).

We discuss now random matrices for which the hypotheses of Theorem 2 are true, and give the form of the corresponding variance and the class of test functions.

(i) Gaussian Ensembles (GOE and GUE), see e.g. [19] for their definitions and properties. Here the measure N is the Wigner semicircle law (12). The CLT for linear statistics with C^1 test functions of compact support is proved in several papers (see e.g. [12] and references therein), and the corresponding variance is

$$V_{1\beta}[\varphi] = \frac{1}{2\beta\pi^2}\int_{-2}^{2} d\lambda \int_{-2}^{2} \left(\frac{\varphi(\lambda)-\varphi(\mu)}{\lambda-\mu}\right)^2 \frac{4-\lambda\mu}{\sqrt{4-\lambda^2}\sqrt{4-\mu^2}} d\mu, \tag{32}$$

where $\beta = 1$ for the GOE and $\beta = 2$ for the GUE. Besides, it follows from the Poincaré inequality for Gaussian matrices [7, 22] that the polynomial C in (21) is $2t^2/\beta$. Thus, assuming that φ of (5) is a smooth enough function decaying sufficiently fast at infinity to have

$$\int_{\mathbb{R}^p} |t_1||t_2||\widehat{\varphi}(t)| d^p t < \infty, \tag{33}$$

we find from Theorem 2 that the CLT for $\mathcal{N}_{pn}[\varphi]/n^{p-1}$ (5) is valid and the corresponding variance is

$$V_{p\beta}[\varphi] = V_{1\beta}[\varphi_p^*], \tag{34}$$

provided that φ_p^* is not identically constant in $[-2, 2]$. It is natural to view the last condition as that describing the "generic" situation. We do not consider here in detail non-generic situations that may lead to different normalizations and limiting laws.

Here is a simple example. Let $p = 2$ and $\varphi(\lambda_1, \lambda_2) = \psi(\lambda_1)\psi(\lambda_2)$, where ψ is a bounded C^1 function that is not identical constant for $|\lambda| \le 2$ and

$$\int_{-2}^{2} \psi(\lambda)\sqrt{4-\lambda^2} d\lambda = 0. \tag{35}$$

Hence, $\varphi_2^\star = 0$ and $V_{2\beta}[\varphi] = 0$. Note however that we have here

$$\mathcal{N}_{2n}[\varphi] = \left(\sum_{l=1}^{n} \psi\left(\lambda_l^{(n)}\right)\right)^2 = (\mathcal{N}_{1n}[\psi])^2 .$$

Since $\mathbf{E}\{\mathcal{N}_{1n}[\psi]\} = 0$ in view of (35) and since the CLT is valid for linear eigenvalue statistics of the GOE and the GUE matrices with bounded C^1 test function, we conclude that in this case $\mathcal{N}_{2n}[\varphi]$ itself, i.e., without the normalizing factor n^{-1} as in (22) for $p = 2$, converges in distribution to the square of the Gaussian random variable with zero mean and the variance $V_{1\beta}[\psi]$.

Analogous results are valid for (4).

Let us show now that the CLT for (4) and (5) is valid with the same variance (34) for bounded functions in $\mathbb{R}^p$ such that $\partial\varphi/\partial\lambda_1$ is bounded in $\mathbb{R}^p$ and is continuous on any compact set of $\mathbb{R}^p$. To this end we introduce the cube $C_A \subset \mathbb{R}^p$ centered in the origin and of the side length A for any $A > 2$, and write

$$\varphi = \varphi_A + \psi_A, \tag{36}$$

where supp $\varphi_A \subset C_A$ and $\partial\varphi_A/\partial\lambda_1$ is continuous in C_A, and supp $\psi_A \subset \mathbb{R}^p \setminus C_{A-1}$ and $\partial\varphi_A/\partial\lambda_1$ is bounded (and locally continuous).

Representation (6) and the Poincaré inequality for Gaussian matrices [7, 22] imply that

$$\begin{aligned}\mathbf{Var}\{n^{-p+1}\mathcal{N}_{pn}[\varphi]\} &\le 2\mathbf{E}\{n^{-p}\mathrm{Tr}\varphi_1(M^{\otimes p})\varphi_1^*(M^{\otimes p})\} \\ &= 2\int_{\mathbb{R}^p} |\varphi_1(\lambda)|^2 \prod_{q=1}^{p} \overline{N}_n(d\lambda_q),\end{aligned} \tag{37}$$

where $\varphi_1(\lambda) = \partial\varphi/\partial\lambda_1$. Let $\{\varphi_k\}$ be sequence of sufficiently smooth functions whose support is in C_A (hence by Theorem 2 the CLT for (5) is valid for every φ_k) and such that

$$\lim_{k\to\infty}\left(\sup_{\lambda\in C_A}|\varphi_A(\lambda)-\varphi_k(\lambda)| + \sup_{\lambda\in C_A}\left|\frac{\partial\varphi_A}{\partial\lambda_1}-\frac{\partial\varphi_k}{\partial\lambda_1}\right|\right) = 0. \tag{38}$$

We have from (36) and (37)

$$\begin{aligned}&\mathbf{Var}\{n^{-p+1}\mathcal{N}_{pn}[\varphi-\varphi_k]\}\\ &\quad\le 4\int_{C_A}\left|\frac{\partial\varphi_A}{\partial\lambda_1}-\frac{\partial\varphi_k}{\partial\lambda_1}\right|^2\prod_{q=1}^{p}\overline{N}_n(d\lambda_q) + 4\int_{\mathbb{R}^p\setminus C_{A-1}}\left|\frac{\partial\psi_A}{\partial\lambda_1}\right|^2\prod_{q=1}^{p}\overline{N}_n(d\lambda_q).\end{aligned} \tag{39}$$

The second term on the r.h.s. is bounded by

$$C_\varphi\int_{|\lambda|>A-1}\overline{N}_n(d\lambda). \tag{40}$$

Denote

$$Z_n = \mathbf{E}\big\{\exp\{ixn^{-p+1}\mathcal{N}^\circ_{pn}[\varphi]\}\big\},\qquad Z_{nk} = \mathbf{E}\big\{\exp\{ixn^{-p+1}\mathcal{N}^\circ_{pn}[\varphi_k]\}\big\},$$

and

$$Z_k = e^{-x^2V_{p\beta}[\varphi_k]/2},\qquad Z = e^{-x^2V_{p\beta}[\varphi]/2}.$$

Then by using an analog of (25) and the Schwarz inequality we have:

$$\begin{aligned}|Z-Z_n| &\le |Z_n-Z_{nk}|+|Z_{nk}-Z_k|+|Z_k-Z|\\ &\le |x|\mathbf{Var}^{1/2}\{n^{-p+1}\mathcal{N}_{pn}[\varphi-\varphi_k]\}+|Z_{nk}-Z_k|+|Z_k-Z|.\end{aligned} \tag{41}$$

The limit $n\to\infty$ in this bound replaces by zero the second term on the right (recall that the CLT valid for every φ_k) and replaces N_n by N in the bound (39)–(40) of the first term (recall that N_n converges weakly to N). The subsequent limit $k\to\infty$ replaces by zero the third term of (41) ($V_{p\beta}$ of (34) and (32) is continuous in the metric, determined by the expression under the "lim" sign in (38)), and replaces by 0 the first term on the right of (39) (in view of (38)). Finally, the limit $A\to\infty$ replaces by zero the r.h.s. of (40) with $\overline{N}_n$ replaced by N.

We obtain that the characteristic function Z_n of $n^{-p+1}\mathcal{N}^\circ_{pn}[\varphi]$ converges to Z, the characteristic function of the Gaussian law with mean zero and variance $V_{p\beta}[\varphi]$, where $\varphi:\mathbb{R}^p\to\mathbb{R}$ is bounded symmetric, $\partial\varphi/\partial\lambda_1$ is bounded in $\mathbb{R}^p$, and such that φ^*_p is not identically constant in $[-2,2]$ (generic situation for the CLT).

Analogous result is valid for $\mathcal{U}_{pn}[\varphi]$ of (4).

(ii) The Wishart and the Laguerre Ensembles (see [10] for their definitions and properties). Here similar results can be obtained by using an argument similar to that for the Gaussian Ensembles, however, the variance of the corresponding GLT for linear statistics is

$$\frac{1}{2\pi^2\beta}\int_{a_-}^{a_+}d\lambda\int_{a_-}^{a_+}\left(\frac{\varphi_p(\lambda)-\varphi_p(\mu)}{\lambda-\mu}\right)^2\frac{4c-(\lambda-a_m)(\mu-a_m)}{\sqrt{4c-(\lambda-a_m)^2}\sqrt{4c-(\mu-a_m)^2}}\,d\mu, \tag{42}$$

where $a_\pm = (1\pm\sqrt{c})^2$, $a_m = c+1$, $c = \lim_{n\to\infty} m/n \in [1,\infty)$.

(iii) Wigner Ensembles (see (ii) of previous section). Here (17) is known since the long time (see [2, 11, 21]). If

$$\sup_n \max_{1\le j,k\le n} E\{(W_{jk}^{(n)})^6\} < \infty,$$

then relation (19) is also valid with $C(t) = A(1+|t|^5)^2$. The CLT for linear statistics is valid for test functions, satisfying

$$\int_{\mathbb{R}} (1+|t|^5)|\widehat{\varphi}(t)|dt < \infty,$$

and with the variance

$$V_{1\beta} + \frac{\kappa_4}{2\pi^2}\left(\int_{-2}^{2} \varphi(\lambda)\frac{2-\lambda^2}{\sqrt{4-\lambda^2}}\right)^2,$$

where $V_{1\beta}$ is defined in (32) and κ_4 is the fourth cumulant of $W_{jk}^{(n)}$, assumed to be independent of j, k, and n [18]. Thus, Theorem 2 implies the validity of the CLT for multilinear statistics with test functions, satisfying

$$\int_{\mathbb{R}^p} (1+|t_1|^5)(1+|t_2|^5)|\widehat{\varphi}(t)|d^pt < \infty.$$

(iv) Sample covariance matrices. By using [18], it is possible to prove results, analogous to those for the Wigner matrices, given in (iii).

(v) Hermitian matrix models, given by (14)–(16) with $\beta = 2$. The corresponding limiting measure N is a unique minimizer of a certain variational problem and has a compact support (see e.g. Theorem 2.1 of [24]). Condition (17) follows from Proposition 2.1 and Theorem 2.1(ii) of [24]. To prove (19) we use the formula

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = \frac{1}{2}\int_{\mathbb{R}^2} (\varphi(\lambda)-\varphi(\mu))^2 K_n^2(\lambda,\mu)d\lambda d\mu \tag{43}$$

for the variance of linear statistics (1) of hermitian matrix models. The formula is an easy consequence of the determinant formulas for the marginals of the joint probability law of these random matrices (see e.g. [19], Sect. 6.2), where K_n is the reproduce kernel of orthogonal polynomials with respect to the weight e^{-nV}. The case of (18) corresponds to $\varphi(\lambda) = e^{it\lambda}$ in (43), hence

$$\mathbf{Var}\{u_n(t)\} \le \frac{t^2}{2}\int_{\mathbb{R}^2} (\lambda-\mu)^2 K_n^2(\lambda,\mu)d\lambda d\mu.$$

Now, Lemma 3.1 of [24] implies that the integral on the r.h.s. is uniformly bounded in n, hence the polynomial C in (19) is At^2 with a n-independent A. The CLT for linear eigenvalue statistics of the hermitian matrix model was proved in [13] for polynomial V in (14), such that the support of the limiting measure N is a connected interval and the variance of the limiting Gaussian law is (32) (see also [23] for more general cases). It follows then from Theorem 2 that the CLT for multilinear statistics is valid for test function, satisfying (33) and with variance (34), where φ_p^* is given by (24).

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